

The cohomology of the weak stable foliation of geodesic flows

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1 Introduction

Let $M = SL(2, R)/\Gamma$ where Γ is a cocompact lattice and GA be the subgroup of upper-triangular matrices. GA is isomorphic of the group of orientation-preserving affine transformations of the real line. The Lie algebra $\mathfrak{sl}(2, R)$ of $SL(2, R)$ has the canonical basis

$$Y = \frac{1}{2} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad S = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad U = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \quad (1.1)$$

satisfying the structural equations

$$[Y, S] = S, \quad [Y, U] = -U \quad \text{and} \quad [S, U] = 2Y \quad (1.2)$$

and for the dual basis $\{\sigma, \eta, \mu\}$ we have

$$d\sigma = \sigma \wedge \eta, \quad d\eta = 2\mu \wedge \sigma \quad \text{and} \quad d\mu = \eta \wedge \mu. \quad (1.3)$$

The right action of $SL(2, R)$ on itself restricts to GA giving the homogeneous action

$$A: M \times GA \rightarrow M, (s, g) = sg \quad (1.4)$$

and the orbit foliation \mathcal{F} is the *weak stable foliation* of the Anosov flow generated by Y . We have the injective homomorphism

$$i_A: \mathcal{L} \rightarrow \chi^\infty(\mathcal{F}) \quad (1.5)$$

of the Lie algebra \mathcal{L} of GA into the Lie algebra $\chi(\mathcal{F})$ of the smooth vector fields tangent to the weak stable foliation \mathcal{F} of the geodesic flow φ_t generated by Y defined by

$$i_A(E)(x) = DA_x(0) \cdot E = E_A(x).$$

The map I_A is defined by

$$I_A: \Lambda_R(\mathcal{L}) \rightarrow \Lambda(\mathcal{F}), \quad I_A(w) \cdot E_A = w_A(E_A) = w(E) \quad (1.6)$$

where $\Lambda_R(\mathcal{L})$ is the exterior algebra of the invariant forms of GA into the leafwise forms on M . Associated to the A -invariant volume form $\sigma \wedge \eta \wedge \mu$ we have the chain map $[S]$

$$P_A: \Lambda(M) \rightarrow \Lambda_R(\mathcal{L}) \quad (1.7)$$

given by

$$P_A(w) \cdot E = \int_M w(E_A) d\nu, \quad E \in \mathcal{L}$$

where $d\nu$ is the measure given by $\sigma \wedge \eta \wedge \mu$, and similarly if w is a 2-form. Since $I(F) \subset \ker P_A$ we have the chain map $P_A^0: \Lambda(F) \rightarrow \Lambda(L)$

The kernel K of P_A^0 is a chain complex and we have the split short exact sequence

$$0 \rightarrow K^j \xrightarrow{i} \Lambda^j(\mathcal{F}) \xrightleftharpoons[i_A]{P_A^0} \Lambda^j(\mathcal{L}) \rightarrow 0 \quad (1.8)$$

giving

$$H^j(\mathcal{F}) = H^j(\mathcal{L}) \oplus H^j(K), \quad 0 < j \leq 2. \quad (1.9)$$

The cohomology of the Lie algebra \mathcal{L} is given by

$$H^1(\mathcal{L}) = R[\eta] \quad \text{and} \quad H^2(\mathcal{L}) = 0. \quad (1.10)$$

2 The computation of the cohomology $H^*(\mathcal{F})$

The main result of this paper is the computation of $H^2(\mathcal{F})$. The computation of $H^1(\mathcal{F}) = R[\eta] \oplus H^1(M, R)$ was done by S. Matsumoto and Y. Mitsumatsu [M.M].

The *reduced leafwise cohomology* $\mathcal{H}^j(\mathcal{F})$ of a foliated manifold (M, \mathcal{F}) is the quotient of the leafwise closed j -forms by the closure of the coboundary $B^j(\mathcal{F})$.

J. Alvares and G. Hector [A.H] have given sufficient conditions for a foliation to have infinite dimension reduced cohomology and gave plenty of examples. S. Matsumoto [M] proved that for the weak stable foliation \mathcal{F} of a geodesic flow we have

$$\mathcal{H}^2(\mathcal{F}) \simeq H^2(M, R). \quad (2.1)$$

Theorem. *If \mathcal{F} is the weak stable foliation of a geodesic flow, then*

$$H^2(\mathcal{F}) = H^2(M, R).$$

Proof. In view of [M., Theorem 13] we have to show that the coboundary space $B^2(\mathcal{F})$ is closed. We recall the definition of leafwise forms. Let $\Lambda(M)$ be the space of smooth differential forms on M and $I(\mathcal{F})$ be the annihilating ideal of \mathcal{F} and $\Lambda(\mathcal{F})$ be the space of the smooth leafwise forms of \mathcal{F} . $\Lambda(\mathcal{F})$ is by definition the quotient space (with the quotient topology)

$$\Lambda^*(M) \xrightarrow{r} \frac{\Lambda^*(M)}{I(\mathcal{F})} = \Lambda^*(\mathcal{F}). \quad (2.2)$$

Thus the projection r is continuous and *open map*. Notice that $B^2(\mathcal{F})$ is closed iff

$$r^{-1}(B^2(\mathcal{F})) = B^2(M) + I^2(\mathcal{F})$$

is closed. The if direction follows from continuity of r . To show the converse notice that $B^2(M) + I^2(\mathcal{F})$ is closed iff

$$\mathcal{A} = \Lambda^2(M) - (B^2(M) + I^2(\mathcal{F}))$$

is open, thus

$$r(\mathcal{A}) = \Lambda^2(\mathcal{F}) - B^2(\mathcal{F}) \quad (2.3)$$

is open, since r is linear and open. Thus $B^2(\mathcal{F})$ is closed. Let $D^2(M, \mathcal{F}) = B^2(M) \cap I^2(\mathcal{F})$ and we will show that

$$\Omega = B^2(M) + I^2(\mathcal{F}) \quad (2.4)$$

is closed in $\Lambda^2(M)$. For this we show first that $I^2(\mathcal{F})$ is closed in Ω . In fact $\Lambda^2(M) - I^2(\mathcal{F})$ is open in $\Lambda^2(M)$ since $I^2(\mathcal{F})$

is closed in $\Lambda^2(M)$. Now $\Omega - I^2(F) = \Omega \cap (\Lambda^2(M) - I^2(F))$ is open in Ω thus $I^2(F)$ is closed in Ω . Now consider

the restriction $P_A^* : \Omega \rightarrow \Lambda^2(L^*)$ of P_A to Ω . The kernel $\ker P_A^*$ is given by

$$\ker P_A^* = \{\theta = d\alpha + \lambda\Lambda\mu, P_A(\theta) = P_A(d_f\alpha) = 0\} \quad (2.5)$$

is closed in $\Lambda^2(M)$. For if $\theta_n \in \ker P_A^*$ converges in the C^∞ topology to $\theta \in \Lambda^2(M)$ then $\theta \in \ker P_A^*$. In fact if $\theta_n = d\alpha_n + \lambda_n\Lambda\mu$

where $\alpha_n = f_n\eta + g_n\sigma + h_n\mu$ then $P_A^*(\theta_n) = P_A^*(d\alpha_n) = P_A^*(d_f\alpha_n) = dP_A(\alpha_n) = 0$ then $P_A(\alpha_n) = c_n\eta$ since $d\eta = 0$ and $d\sigma = \sigma\Lambda\eta$

in $\Lambda(L^*)$. Now since θ_n converges to θ in the C^∞ topology then $P_A^*(\theta_n) = dP_A(\alpha_n) = c_n d\eta$ converges in the C^∞ topology to $P_A^*(\theta) = P_A(d\alpha) =$

$cd\eta = 0$ where $I_A(c\eta) = \alpha$ thus $\theta = d\alpha + \lambda\Lambda\mu$ belongs to Ω . Now I claim that

$$\Omega = \ker P_A^* + R[d\sigma_A] \quad (2.6)$$

For if $\theta \in \Omega$ then $\theta = d\alpha + \lambda\Lambda\mu$ where $\alpha = f\eta + (g+c)\sigma + h\mu$ and we may assume that $\int_M g d\nu = 0$ where $d\nu$ is the measure given by

the canonical volume form $\sigma\Lambda\eta\Lambda\mu$. So $P_A^*(\theta) = P_A^*(d\alpha) = P_A^*(d_f\alpha) = \left\{ \int_M (Sf - Yg + (g+c)d\nu) d\sigma = cd\sigma \right.$ so

$$\theta = \theta_0 + \lambda\Lambda\mu + cd\sigma_A$$

where $\theta_0 = d(f\eta + g\sigma + h\mu) + \lambda\Lambda\mu$ belongs to $\ker P_A^*$ proving that $\Omega = \ker P_A^* + R[d\sigma_A]$. Thus Ω is closed since $\ker P_A^*$ is closed and $R[d\sigma_A]$ is finite dimensional.

3 The computation of $D^j(M, \mathcal{F})$, $2 \leq j \leq 3$

In [S] we considered the cohomology $H^j(M, \mathcal{F})$ of the complex $(I(\mathcal{F}), d)$ and the distinguished closed subspaces $D^j(M, \mathcal{F}) = B^j(M) \cap I(\mathcal{F})$. From the short exact sequence

$$0 \rightarrow I(\mathcal{F}) \xrightarrow{i} \Lambda(M) \xrightarrow{r} \Lambda(\mathcal{F}) \rightarrow 0 \quad (3.1)$$

we have the long exact cohomology sequence

$$\begin{aligned} 0 \rightarrow H^1(M, R) &\xrightarrow{r} H^1(\mathcal{F}) \xrightarrow{\delta} D^2(M, \mathcal{F}) \\ &\rightarrow H^2(M, R) \xrightarrow{r} H^2(\mathcal{F}) \xrightarrow{\delta} D^3(M, \mathcal{F}) \rightarrow 0. \end{aligned} \quad (3.2)$$

Now from $H^1(\mathcal{F}) = R[\eta] \oplus H^1(M, R)$ and $H^2(\mathcal{F}) = H^2(M, R)$ and (3.2) we see that

$$D^2(M, \mathcal{F}) = R[d\eta] \quad \text{and} \quad D^3(M, \mathcal{F}) = 0. \quad (3.3)$$

Remark. For any smooth function $f: M \rightarrow R$ the 2-form $f\sigma \wedge \eta$ extends to a closed 2-form $w = f\sigma \wedge \eta + g\sigma \wedge \mu + h\eta \wedge \mu$ i.e. the PDE $Uf = Yg - Sh$ has a smooth solution (g, h) for each smooth function $f: M \rightarrow R$. This follows from (3.2) and $D^3(M, \mathcal{F}) = 0$.

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